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Impact of Linear Homogeneous Recurrent Relation Analysis

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ABSTRACT

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence. A recurrence relation is an equation that recursively defines a sequence or multidimensional array of values, once one or more initial terms are given; each further term of the sequence or array is defined as a function of the preceding terms.

1. INTRODUCTION

A recurrence relation is an equation that defines a sequence based on a rule that gives the next term as a function of the previous term(s). The simplest form of a recurrence relation is the case where the next term depends only on the immediately previous term. If we denote the nnth term in the sequence by xnxn, such a recurrence relation is of the form xn+1=f(xn)xn+1=f(xn) for some function ff. One such example is xn+1=2−xn/2.xn+1=2−xn/2. A recurrence relation can also be higher order, where the term xn+1xn+1 could depend not only on the previous term xnxn but also on earlier terms such as xn−1xn−1, xn−2xn−2, etc. A second order recurrence relation depends just on xnxn and xn−1xn−1 and is of the form xn+1=f(xn,xn−1)xn+1=f(xn,xn−1) for some function ff with two inputs. For example, the recurrence relation xn+1=xn+xn−1xn+1=xn+xn−1 can generate the Fibonacci numbers. To generate sequence basd on a recurrence relation, one must start with some initial values.

For a first order recursion xn+1=f(xn)xn+1=f(xn), one just needs to start with an initial value x0x0 and can generate all remaining terms using the recurrence relation. For a second order recursion xn+1=f(xn,xn−1)xn+1=f(xn,xn−1), one needs to begin with two values x0x0 and x1x1. Higher order recurrence relations require correspondingly more initial values.

A linear **homogeneous** recurrence of order kk is expressed this way: A0an+A1an−1+A2an−2+⋯+Akan−k=0

2. Objectives

The aim of this paper is to solve the linear recurrence relation

$x_{n+1} = a_0 x_n + a_1 x_{n-1} + \cdots + a_{n-1} x_1 + a_n x_0$, n = 0, 1, 2, . . . ,

when its constant coefficients are in arithmetic, respective geometric progression. Rather surprising, when the coefficients are in arithmetic progression, the solution is a sequence of certain generalized Fibonacci numbers, but not of usual Fibonacci numbers, while if they are in geometric progression the solution is again a geometric progression, with different ratio. Recurrence relations con be divided into two: Linear and Non-Linear. Linear homogeneous recurrence relations with constant coefficients; Solving linear homogeneous recurrence relations with constant coefficients; Solving linear homogeneous recurrence relations with constant coefficients of degree two and degree three; Generating functions; Using generating functions to solve recurrence relations. Some recurrence relations are solvable using algebraic techniques, but they're often tricky, they require some math many of you don't know, and it still won't work for many relations.

3. Methodology

Linear Homogeneous Recurrence Relations Definition: A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form a n = c 1 a n −1 + c 2 a n −2 + ….. + c k a n − k, where c 1, c 2, ….,c k are real numbers, and c k ≠ 0 it is linear because the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of n. it is homogeneous because no terms occur that are not multiples of the a j s. Each coefficient is a constant. The degree is k because a n is expressed in terms of the previous k terms of the sequence. By strong induction, a sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the k initial conditions a 0 = C 0, a 0 = C 1, …, a k −1 = C k −1. Examples of Linear Homogeneous Recurrence Relations P n = (1.11) P n-1 linear homogeneous recurrence relation of degree one f n = f n-1 + f n-2 linear homogeneous recurrence relation of degree two not linear H n = 2 H n −1 + 1 not homogeneous B n = n B n −1 coefficients are not constants. Solving Linear Homogeneous Recurrence Relations The basic approach is to look for solutions of the form a n = r n, where r is a constant. Note that a n = r n is a solution to the recurrence relation a n = c 1 a n −1 + c 2 a n −2 + ⋯ + c k a n − k if and only if r n = c 1 r n −1 + c 2 r n −2 + ⋯ + c k r n − k. Algebraic manipulation yields the characteristic equation: r k − c 1 r k −1 − c 2 r k −2 − ⋯ − c k −1 r − c k = 0 The sequence {a n } with a n = r n is a solution if and only if r is a solution to the characteristic equation. The solutions to the characteristic equation are called the characteristic roots of the recurrence relation. The roots are used to give an explicit

formula for all the solutions of the recurrence relation. Solving Linear Homogeneous Recurrence Relations of Degree Two:

Theorem 1: Let c 1 and c 2 be real numbers. Suppose that r 2 − c 1 r − c 2 = 0 has two distinct roots r 1 and r 2. Then the sequence {a n } is a solution to the recurrence relation a n = c 1 a n −1 + c 2 a n −2 if and only if for n = 0, 1, 2,..., where α 1 and α 2 are constants.

Example: What is the solution to the recurrence relation a n = a n −1 + 2 a n −2 with a 0 = 2 and a 1 = 7 ? Solution: The characteristic equation is r 2 − r − 2 = 0. Its roots are r = 2 and r = −1. Therefore, {a n } is a solution to the recurrence relation if and only if a n = α 1 2 n + α 2 (−1) n, for some constants α 1 and α 2. To find the constants α 1 and α 2, note that a 0 = 2 = α 1 + α 2 and a 1 = 7 = α 1 2 + α 2 (−1). Solving these equations, we find that α 1 = 3 and α 2 = −1. Hence, the solution is the sequence {a n } with a n = 3·2 n − (−1) n.
To find an explicit formula for the Fibonacci numbers: The sequence of Fibonacci numbers satisfies the recurrence relation f n = f n −1 + f n −2 with the initial conditions: f 0 = 0 and f 1 = 1. Solution: The roots of the characteristic equation r 2 − r − 1 = 0. For some constants α 1 and α 2: Using the initial conditions f 0 = 0 and f 1 = 1, we have Solving, we obtain.

Theorem 2 : Let c 1 and c 2 be real numbers with c 2 ≠ 0. Suppose that r 2 − c 1 r − c 2 = 0 has one repeated root r 0. Then the sequence {a n } is a solution to the recurrence relation a n = c 1 a n −1 + c 2 a n −2 if and only if for n = 0,1,2,..., where α 1 and α 2 are constants.

Example: What is the solution to the recurrence relation a n = 6a n −1 − 9 a n −2 with a 0 = 1 and a 1 = 6? Solution: The characteristic equation is r 2 − 6 r + 9 = 0. The only root is r = 3. Therefore, {a n } is a solution to the recurrence relation if and only if a n = α 1 3 n + α 2 n(3) n where α 1 and α 2 are constants. To find the constants α 1 and α 2, note that a 0 = 1 = α 1 and a 1 = 6 = α 1 · 3 + α 2 ·3. Solving, we find that α 1 = 1 and α 2 = 1. Hence, a n = 3 n + n 3 n.

Solving Linear Homogeneous Recurrence Relations of Arbitrary Degree This theorem can be used to solve linear homogeneous recurrence relations with constant coefficients of any degree when the characteristic equation has distinct roots. Theorem 3 : Let c 1, c 2,..., c k be real numbers. Suppose that the characteristic equation r k − c 1 r k −1 −⋯ − c k = 0 has k distinct roots r 1, r 2, ..., r k. Then a sequence {a n } is a solution of the recurrence relation a n = c 1 a n −1 + c 2 a n −2 + ….. + c k a n − k if and only if for n = 0, 1, 2, ..., where α 1, α 2,..., α k are constants.

4. Method and Solution

$U_n = C_1U_{n-1} + C_2U_{n-2} + \ldots\ldots + C_d\, U_{n-d}$

Where $C_1, C_2, \ldots\ldots, C_d$ are number and $C_d \neq 0$.
$P_n = U_n + U'$ and $q_n = \beta\, U_n$
$P_n = (C_1 U_{n-1} + C_2 U_{n-2} + \ldots\ldots + C_d U_{n-d}) + (C_n U'_n + C_2U'_{n-2} + \ldots..+ C_d U'_{n-d})$
$= C_1 (U_{n-1} + U'_{n-1}) + C_2 (U_{n-2} + U'_{n-2}) + \ldots\ldots + C_d (U_{n-d} + U'_{n-d})$
$= C_1 P_{n-1} + C_2 P_{n-2} + \ldots\ldots + C_d P_{n-d}$

So, P_n is a solution of the recurrence.
$q_n = \beta\, U_n$
$= \beta\, (C_1 U_{n-1} + C_2 U_{n-2} + \ldots.. + C_d U_{n-d})$
$= C_1 (\beta U_{n-1}) + C_2 (\beta U_{n-2}) + \ldots\ldots + C_d (\beta U_{n-d})$
$= C_1 q_{n-1} + C_2 q_{n-2} + \ldots. + Cd\, q_{n-d}$
So, q_n is a solution of recurrence.

Example: What is the solution of recurrence relation.
$U_n = 2 U_{n-1} + U_{n-2} - 2 U_{n-3}, U_0 = 3, U_1 = 6, U_2 = 0$
The Characteristic equation is

$x^n - 2x^{n-1} - x^{n-2} + 2x^{n-3} = 0$
$x^3 - 2x^2 - x + 2 = 0$
$x_1 = -1, x_2 = 2, x_3 = 1$

The general solution is $U_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n$
$U_n = \beta_1(-1)^n + \beta_2 2^n + \beta_3 1^n$

$U_0 = 3 \Rightarrow \beta_1 + \beta_2 + \beta_3 = 3$
$U_1 = 6 \Rightarrow -\beta_1 + 2\beta_2 + \beta_3 = 6$
$U_2 = 0 \Rightarrow \beta_1 + 4\beta_2 + \beta_3 = 0$
$\beta_1 = -2, \beta_2 = -1, \beta_3 = 6$

The solution is $U_n = (-2)(-1)^n + (-1)2^n + 6$

5. Conclusion

A simple and efficient method for solving linear homogeneous recursive relations has been introduced, which mainly involves synthetic divisions. This method has the advantage that it does not need to use generating function techniques or solve a system of linear equations to determine the unknown coefficients of the solution. A variety of techniques is available for finding explicit formulas for special classes of recursively defined sequences. The method explained here works for the Fibonacci and other similarly defined sequences.

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